Error Detecting Codes

Mathematics is everywhere

Mathematics does a lot of hidden work in our technological society. Most people are not aware of the fact that neat abstract mathematics is used to make sure that things work well. In this workshop we will investigate one of the ways that hidden mathematics allows us to trust in communication.

Codes are everywhere

- Streaming movies
- Sending images
- Barcodes
- ISBN (International standard book numbers)
- Credit cards
- Medicare numbers
- TFN (Tax file numbers)

All of these items are communicated from one party to another, and it is important that the communication is accurate. In each instance, what are the consequences of miscommunication?

Credit cards

Types of common errors

Error	%	Example
Single digit	79.1	19456 →19486
Transposition of adjacent digits	10.2	19456 → 19546
Jump transposition	0.8	19456 → 19654
Twin	0.5	19455 → 19466
Phonetic	0.5	19160 → 19116
Jump twin	0.3	19156 → 49456

https://www.uu.edu/dept/math/SeniorPapers/01-02/Oldham.pdf

A trick with credit cards

- Consider the credit card number

  5415 1234 5678 1234
- Record the results of 5x2, 4x1, 1x2 , 5x1…(digits in odd places are doubled)
- If a result has two digits, subtract 9 from it to obtain a single digit result instead (equivalently, replace any two digit result by the sum of the digits). Now you have a collection of 16 single digit results.
- Add all of the single digit results together.

The sum (which we shall call the Luhn sum) is a multiple of 10.

Let's try it out!

- Record the results of 1st digit x 2, 2nd digit x 1, 3rd digit x 2 , 4th digit x 1…
- If a result has two digits, subtract 9 from it to obtain a single digit result instead
- Add all of the single digit numbers you have written down
- The result (the Luhn sum) should be a multiple of 10

Luhn's Error Detecting Code in one sentence

A string of 16 digits

$$d_1\ d_2\ \ldots d_{16}$$

is a **valid credit card number** if the Luhn sum is a multiple of 10.

How to make a credit card number

- Choose 6 digits to indicate the institution that issued the card
- Choose 9 digits to indicate which account at that institution is responsible for the card
- Choose the last digit, the check digit, in the only way that you can to obtain a valid credit card number.

If the first 15 digits identify the institution and the account, then what is the point of the 16th digit?

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If we try to communicate a credit card number, and a mistake is made, we can usually tell that a mistake was made because the number received is not valid.

Knowing this, we can simply ask for the number to be retransmitted.

This is an example of an Error Detecting Code.

An error in one digit is definitely detected

Suppose that we mean to write down a credit card number

$$d_1\ d_2\ \ldots\ d_i\ \ldots d_{16}$$

but we make a mistake in the i-th digit. That is, we write down

$$d_1\ d_2\ \ldots\ e_i\ \ldots d_{16}$$

instead. How will the Luhn sum change?

We shall call the Luhn sum corresponding to $d_1\ d_2\ \ldots\ d_i\ \ldots d_{16}$ the *old Luhn sum*, and the Luhn sum corresponding to $d_1\ d_2\ \ldots\ e_i\ \ldots d_{16}$ the *new Luhn sum*.

An error in one digit is definitely detected

Each digit in the credit card number contributes a single digit to the Luhn sum.

For even positions

Digit	0	1	2	3	4	5	6	7	8	9
Contribution to the Luhn sum	0	1	2	3	4	5	6	7	8	9

For odd positions

Digit	0	1	2	3	4	5	6	7	8	9
Contribution to the Luhn sum	0	2	4	6	8	1	3	5	7	9

An error in one digit is definitely detected

When we change d_i to e_i, we change one of the contributions to the Luhn sum, swapping a one digit contribution for a different one digit contribution. The difference between two distinct digits is not a multiple of 10. Thus we change the Luhn sum by an amount that is not a multiple of 10.

Since the old Luhn sum was a multiple of 10, and the difference between the new and the old Luhn sums is not a multiple of 10, the new Luhn sum is not a multiple of 10.

Thus the new number is not a valid credit card number.

ISBN – International Standard Book Number

What are ISBNs?

The 10-digit number has a curious property

The 10-digit number we saw was 0552149519, and

										Sum
0	5	5	2	1	4	9	5	1	9	
×	×	×	×	×	×	×	×	×	×	
1	2	3	4	5	6	7	8	9	10	
0	10	15	8	5	24	63	40	9	90	264

and 264 is a multiple of 11 (because $264 = 24 \times 11$)

Algebraic representation – checksum and check digit

A publication is assigned a 9 digit identication code, which we shall denote

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9$$

VERSION 1: The tenth character a_{10} is the unique choice from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X such that

$$\underbrace{1 \times a_1 + 2 \times a_2 + \ \ldots + 9 \times a_9 + 10 \ \times a_{10}}$$

This is called the checksum

is a multiple of 11.

Calculate the check digit using the checksum approach

At age 17 you have just written your autobiography, “The Greatness of Being Great”. It is getting an ISBN-10. Based on some publication particulars of the book, the nine digit identifying code is to be:

7 8 1 4 5 3 1 2 2 ?

Hint

At age 17 you have just written your autobiography, “The Greatness of Being Great". It is getting an ISBN-10. Based on some publication particulars of the book, the nine digit identifying code is to be:

7 8 1 4 5 3 1 2 2 ?

Calculate the check digit:

1. Compute the sum $1 \times 7 + 2 \times 8 + 3 \times 1 + \cdots + 9 \times 2 + 10 \times D$
2. From the algebraic equation, use trial and error to determine D

Algebraic representation – checksum and check digit

A publication is assigned a 9 digit identication code, which we shall denote

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9$$

VERSION 2: The tenth character a_{10} is the remainder/check digit when

$$1 \times a_1 + 2 \times a_2 + \ \ldots + 9 \times a_9$$

is divided by 11. If the remainder is 10, we write X (it's a Roman thing!).

Calculate the check digit using the check digit approach

At age 17 you have just written your autobiography, “The Greatness of Being Great”. It is getting an ISBN-10. Based on some publication particulars of the book, the nine digit identifying code is to be:

7 8 1 4 5 3 1 2 2 ?

Hint

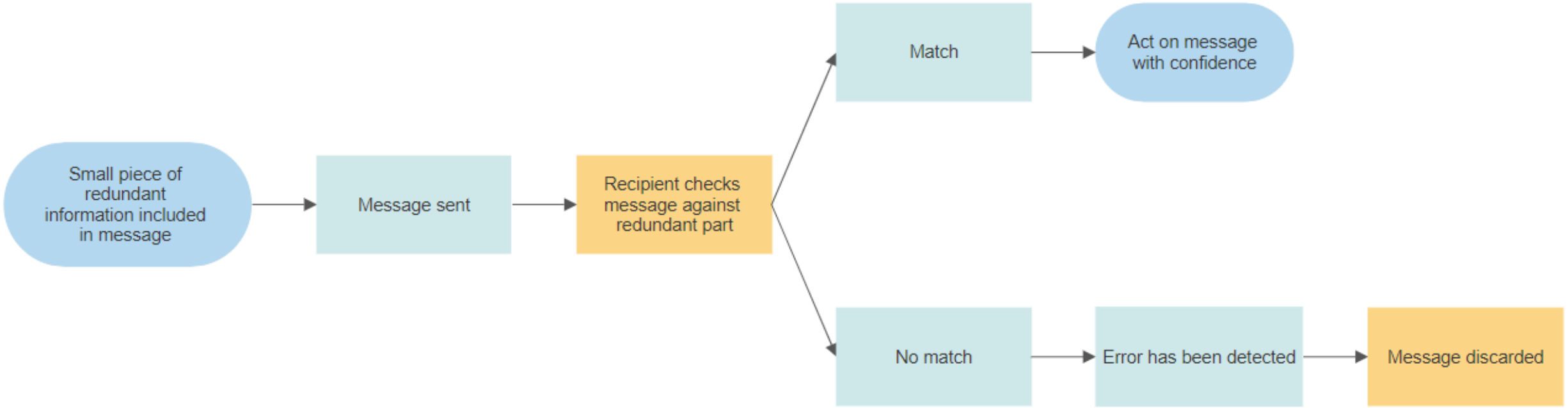
At age 17 you have just written your autobiography, “The Greatness of Being Great". It is getting an ISBN-10. Based on some publication particulars of the book, the nine digit identifying code is to be:

7 8 1 4 5 3 1 2 2 ?

Calculate the check digit:

1. Compute the sum $1 \times 7 + 2 \times 8 + 3 \times 1 + \cdots + 9 \times 2$
2. Divide the sum by 11 and find the remainder
3. The remainder is the check digit.

Application of error detecting codes



Properties of an Excellent Error Detecting Code

An excellent error detecting code should:

- Add only a small amount of redundant information (because space and time are precious resources in communication)
- Guarantee to detect most of the errors that occur.

These are competing criteria, so compromise is necessary.

Some Tasks and Problems

Make your own valid credit card number.

Ask your Mum or Dad or Guardian to show you their credit card, and use it to explain how Luhn's Error Detecting Code works.

Write a short report explaining the benefits of including an error detecting code when you issue identification numbers.

Research how the error detecting code in ISBN-13 works. Compare it to ISBN-10.

More tasks and problems: Respond to the following

Many people say that in mathematics there is always just one correct answer. Is there one correct error detecting code scheme?

What role does mathematics play in “getting things right as efficiently as possible"?

In which ways are mathematicians creative?

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Thank you for participating.
We hope you had fun!

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